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THE SCOPE OF THE IMAGE METHOD

by

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Abstract

The image method is defined and all regions, equations, and combinations of boundary conditions to which it applies are determined.

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I. Introduction

The image method is a method for constructing a Green's function for a part of space bounded by planes in terms of the corresponding Green's function for the full space. The purpose of this note is the determination of all regions, equations and boundary conditions to which this method is applicable.

We begin by considering a linear second-order differential operator L and its formal adjoint M, both defined in a full n-dimensional Euclidean space. We may define a Green's function of the operator L for any sufficiently regular region D and a homogeneous boundary condition on the boundary of D. This is a function $G(r,r')$ of two points in D which satisfies the homogenous equation $MG = 0$ with respect to its second argument r' at all points in D except $r' = r$, at which point it has an appropriate singularity. In addition G satisfies the prescribed boundary condition. If the region D extends to infinity, an appropriate homogenous condition must also be imposed there.

We will now describe the image method for the case in which D is bounded by hyperplanes and the normal derivative vanishes on the boundary of D. First, suppose $G^*(r,r')$ is that Green's function of the operator L for the full space which satisfies given conditions at infinity. Let S(r) be the set of images of r in the bounding planes of D, plus the images of these images in the bounding planes of D, etc. Now we construct the function

$$(1) \qquad G(r,r') = \sum_{r'' \in S(r)} G^*(r'', r') .$$

It is clear that S(r) is symmetric in every bounding plane of D and therefore that the normal derivative of $G(r,r')$ vanishes on the boundaries of D. For certain operators L, each term of the sum in (1) is also a solution of M. Thus for these operators, G will be the Green's function for D, assuming the sum converges, and provided that G has only one singularity in D. This will be the case if S(r) has only one point, r itself, in D.

We can now formulate our problems. The first problem is to find those regions D with the property that the set S(r) has only one point, r itself, in D, for every point r in D. It is clear that these are the only regions for which the image method described above is applicable. The second problem

is that of finding, for each such region D, all those operators L for which each term in (1) is a solution of MG = 0. It is also clear that for each admissible region D, these are the only operators for which the image method just described applies. The third problem concerns the extension of the image method to the boundary conditions $G = 0$ and $\frac{\partial G}{\partial n} + hG = 0$ and combinations of these conditions. This problem will be further explained in Section III.

II. Admissible Regions

A region D for which the image method is applicable we will call admissible. Let R_i denote a reflection in side i of an admissible region D. The set S(r) of images of any point r in D is the set of points obtained by applying to r all the reflections in the group generated by the R_i. If D is admissible, S(r) has only one point, r itself, in D. But this implies that the group generated by the R_i is discrete and that D is a fundamental domain of this group.

However, not every fundamental domain of a discrete group of reflections is an admissible region for the image method. In fact, the angle between any two planes bounding the fundamental domain of a discrete group of reflections is of the form $\frac{2\pi}{p}$, where p is an integer.[1] But if p is odd, then it is easily seen that S(r) will have at least two points in D. To establish this we need only consider successive reflections in the two sides which make the angle $\frac{2\pi}{p}$. Therefore only those fundamental domains of discrete groups of reflections for which the angle between any two bounding planes is of the form $\frac{\pi}{p}$, with p integral, can be admissible regions for the image method. Since all the fundamental domains of discrete groups of reflections are known,[1] it can be immediately verified that in two and three dimensions all those domains which satisfy this additional condition are admissible regions for the image method.

Thus all the admissible regions for the image method have been determined. Those in two and three dimensions are shown in Figures 1 and 2 respectively. Sommerfeld [2] mentions most of these admissible regions in his discussion of the image method.

III. Other Boundary Conditions

If the condition that the normal derivative must vanish is replaced by the condition that the function vanish on some or all of the bounding planes

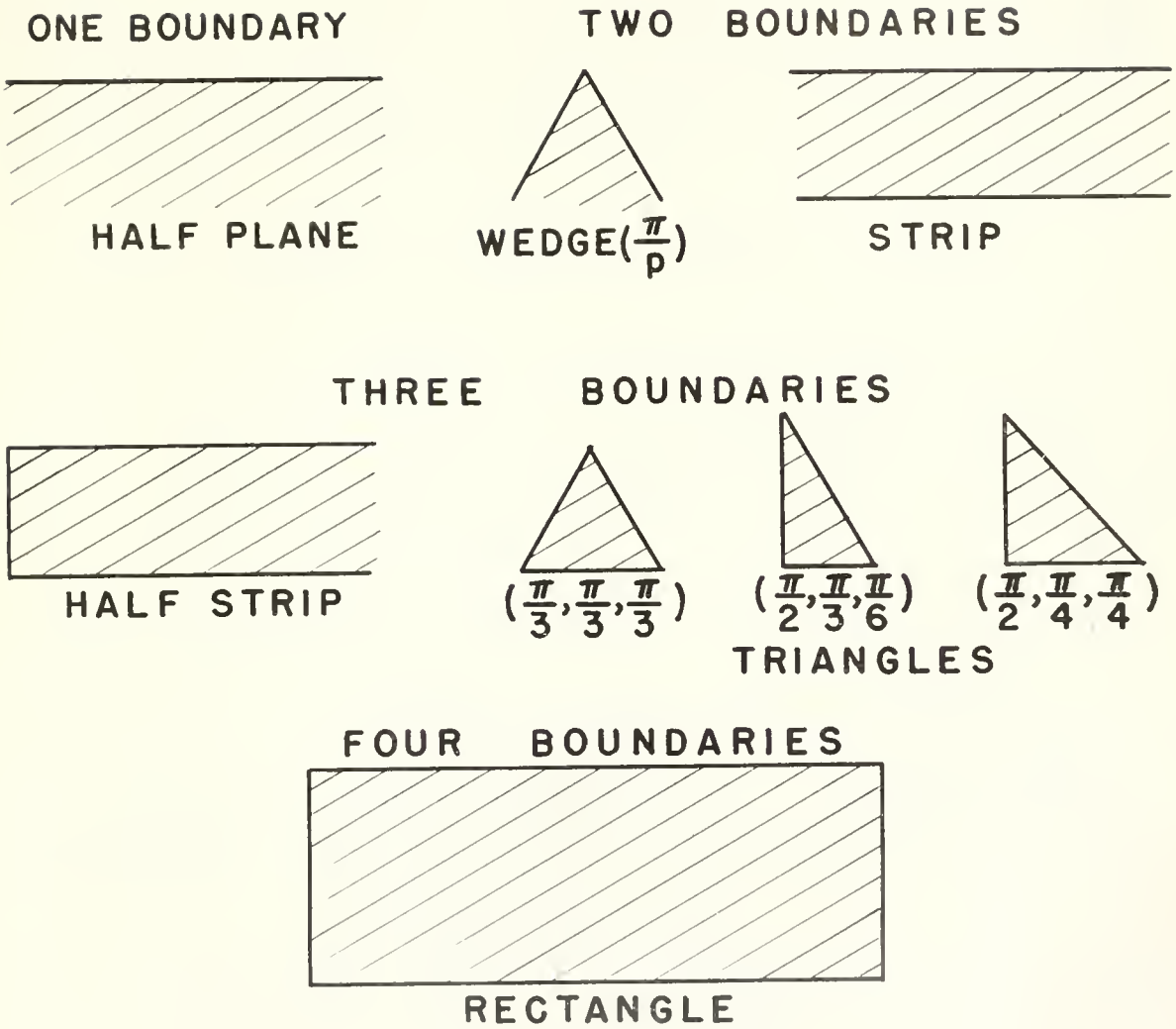

Figure 1

Admissible regions for application of image method; two dimensions

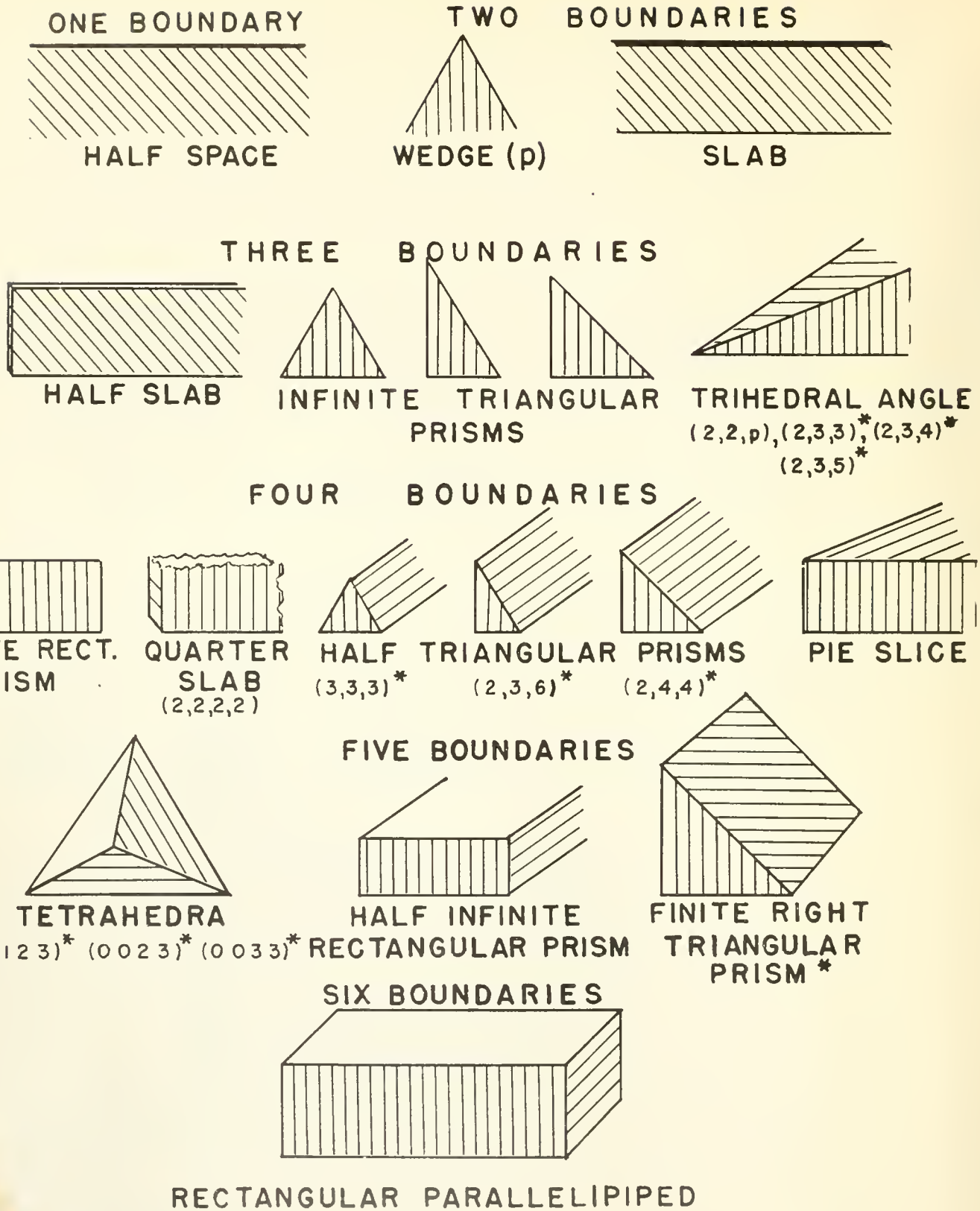

Figure 2

Admissible regions for application of image method; three dimensions.

For regions denoted by*, Laplace's equation is not solvable by separation of variables.

of D, then the image method previously described must be modified. This modification can be made if it is possible to assign a plus or minus sign to each point in S(r) in such a way that two points which are images of each other in the bounding plane B have the same or opposite signs according as the normal derivative or the function vanishes on B. For then we define G by

$$G(r,r') = \sum_{r'' \in S(r)} (\pm)\, G^*(r'', r'). \tag{2}$$

The sign of each term in the sum is that assigned to the point r" in S(r), assuming that r itself has a plus sign. It is clear that (2) yields a function which is odd with respect to each plane on which the boundary condition requires that the function vanish, and thus this condition is fulfilled. Similarly (2) is even with respect to every plane on which it is required that the normal derivative vanish, and therefore this condition is also met.

Thus we see that (2) yields that Green's function for the operator L in the region D which satisfies the imposed boundary conditions, provided that the previous conditions on L and D are satisfied, and provided also that it is possible to assign plus and minus signs to S(r) in the requisite manner. The conditions on L have not yet been examined, but all the admissible regions D have been determined. We now consider the possibility of assigning plus and minus signs.

Each region D is a fundamental domain of a discrete group of reflections which satisfies the extra condition that the angle between any two planes w_i and w_j is of the form $\frac{\pi}{p_{ij}}$. Now it is a known theorem that the discrete group is completely characterized by the relations

$$R_i^2 = 1, \quad (R_i R_j)^{p_{ij}} = 1. \tag{3}$$

(When two walls are parallel, p_{ij} is infinite and the corresponding relation is to be disregarded.) We make use of this result by supposing that the boundary condition G = 0 is imposed on the planes $w_1 \ldots w_S$ and $\frac{\partial G}{\partial n} = 0$ is imposed on the remaining planes $w_{S+1} \ldots w_K$. Now we consider the group generated by $-R_1, \ldots, -R_S, R_{S+1}, \ldots R_K$. We assign a plus or minus sign to each image according as it is obtained from r by a reflection bearing a plus or minus sign.

In order that this assignment be unique it is obviously necessary and sufficient, in virtue of the above theorem (see eq. (3)), that

$$(4) \qquad (R_i' R_j')^{p_{ij}} = 1.$$

Here R_i' and R_j' stand for plus or minus R_i and R_j respectively, according as R_i and R_j appear with plus or minus signs in the generators of the group. Thus we arrive at the result that the desired Green's function can be constructed by the image method if and only if p_{ij} is even for every pair of walls on which different boundary conditions are imposed. The combinations of boundary conditions which can be employed with each admissible region are thus completely determined. In particular, if the function is to vanish on every bounding plane then no additional condition on the angles is required.

It is also possible to extend the image method to cases in which the boundary condition on some or all bounding planes is

$$(5) \qquad \frac{\partial G}{\partial n} + hG = 0.$$

Here h is a constant for each plane and may be different for different planes. When (5) holds, we must introduce line, surface, and volume images, and so on for higher dimensions. First we will give the result for the case of a single plane, which we take to be $x' = 0$. Omitting all variables except x, we have

$$(6) \qquad G(x,x') = G^*(x,x') + G^*(-x,x') + 2h \int_{\infty}^{-x} e^{h(x+\eta)} G^*(\eta,x')d\eta$$

It is easily verified that (6) satisfies (5) on $x' = 0$ provided that

$$(7) \qquad \frac{\partial G^*}{\partial x}(x,x') = -\frac{\partial G^*}{\partial x'}(x,x').$$

Thus (6) is the Green's function for the half-space bounded by the plane $x' = 0$ and satisfies (5) on this plane.

To obtain the Green's function for any region D bounded by planes on each of which $G = 0$ or $\frac{\partial G}{\partial n} = 0$ or $\frac{\partial G}{\partial n} + hG = 0$, we proceed as follows. To each bounding plane we assign a generalized reflection operator. If $\frac{\partial G}{\partial n} = 0$ on a plane, the corresponding operator is the ordinary reflection operator,

while if G = 0 on a plane, the operator is the ordinary reflection operator multiplied by -1. However, if the condition (5) holds on a plane, the corresponding operator assigns to each point its ordinary image plus a line of images on the half-ray which is normal to the plane and extends from the image point away from the plane. The ordinary image is assigned the same strength as the antecedent point. Each point on the ray is assigned a strength equal to $2he^{hd}$ times the strength of the antecedent point. The strength of the original point r is +1, that of an image in a plane on which $\frac{\partial G}{\partial n} = 0$ is also +1, and that in a plane on which G = 0 is -1. In this way all the image points and their strengths are defined.† Now the Green's function G(r,r') for the region D is constructed by forming the sum $\sum_{r''} G^*(x'',x')\, A(r'')$, where the sum extends over all images of r and where A(r'') is the strength of each image. The sum will, of course include line, surface, and volume integrals, etc.

In order for this construction to succeed it is again necessary that no image lie within D, except r itself. Therefore D must be one of the regions previously determined. If the boundary condition on some planes is that G = 0, then the angles between these planes and the others must be even submultiples of π. The 'other' planes are those on which $\frac{\partial G}{\partial n} = 0$ or $\frac{\partial G}{\partial n} + hG = 0$. Furthermore, in order that no line of images lie in D it is necessary that the angle between any plane on which (5) is imposed and any other plane be $\frac{\pi}{2}$, as is easily verified.

These results completely characterize the regions and combinations of boundary conditions to which the extended image method applies.

IV. Admissible Differential Equations

In order that the image method apply, the only requirement on the linear differential operator L is that each term in the sum (1), (2), or in the sum described at the end of the previous section also be a solution of MG = 0. The necessary and sufficient condition for this to be the case is that M admit the group of reflections in the boundaries of D, i.e., that M be invariant under these reflections. In the case of the mixed boundary condition, (7) must also hold. These considerations determine the differential equations to which the image method applies.

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† Iteration of the generalized reflection operation is defined to be the identity.

As an example, let us consider operators M in two variables with constant coefficients. The equation Mu = 0 is equivalent to

$$(8)\quad au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0.$$

If x', y' is the image of x,y in a line making the angle θ with the x-axis, then (8) becomes in the x', y' coordinates

$$(9)\quad (a\cos^2 2\theta + b \sin 2\theta \cos 2\theta + c \sin^2 2\theta)\, u_{x'x'}$$
$$+ (a \sin^2 2\theta - b \sin 2\theta \cos 2\theta + c \cos^2 2\theta) u_{y'y'}$$
$$+ (2a \sin 2\theta \cos 2\theta + b \sin^2 2\theta - b \cos^2 2\theta - 2c \sin 2\theta \cos 2\theta) u_{x'y'}$$
$$+ (d \cos 2\theta + e \sin 2\theta) u_{x'} + (d \sin 2\theta - e \cos 2\theta) u_{y'} + fu = 0.$$

In order that (8) be invariant under this reflection, (9) must coincide with (8). Therefore the coefficients in (9) must be multiples of the corresponding coefficients in (8). If f is not zero, then the multiple is one.

We thus obtain a set of five linear homogeneous equations for the coefficients a to e, which separate into a set of three for a,b,c and a set of two for d, e. The rank of each coefficient matrix in these equations is one, so that the first three equations have a two-parameter set of solutions, and the last two a one-parameter set. They are given by

$$(10)\qquad \frac{b}{a-c} = \tan 2\theta$$

$$(11)\qquad \frac{e}{d} = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

If both b = 0 and a - c = 0, (10) is unnecessary and the first three equations for a, b, c hold for every value of θ.

The first result is then, that if $f \neq 0$, (8) is invariant under reflection only for those angles θ which satisfy (10) and (11). If both b and a - c are zero, (10) is unnecessary, while if both d and e are zero, (11) is unnecessary.

In the interval $0 \leq \theta < \pi$, (11) has only one solution and (10) has only two which differ by $\frac{\pi}{2}$. Therefore when (10), (11) apply there is at most one angle θ satisfying both. If (11) doesn't apply but (10) does, there are exactly two angles, differing by $\frac{\pi}{2}$, which satisfy (10). If neither (10) nor (11) applies then θ is unrestricted. Thus we have the following results provided $f \neq 0$.

$e = d = b = 0,\ a = c$	all θ
e, d not both zero, $b = 0,\ a = c$	one θ given by (11)
e, d not both zero; b, a - c not both zero	at most one θ given by (10, 11)
$e = d = 0$; b, a - c not both zero	two angles θ, $\theta + \frac{\pi}{2}$ given by (10),

If $f = 0$, then in addition to the above results we find that if $a = -c$, other possible values of θ are roots of

$$\tan 2\theta = \frac{2a}{b}\ ,\quad \frac{\sin 2\theta}{1 + \cos 2\theta} = -\frac{d}{e}\ .$$

Again the first has two roots differing by $\frac{\pi}{2}$ and the second only one root. Thus if both apply there is only one value of θ, and if only the first applies there are two values of θ differing by $\frac{\pi}{2}$.

V. Conclusion

The image method has been defined for regions in Euclidean n-space bounded by planes, and all the regions to which it applies have been determined for every choice of boundary conditions. The boundary conditions considered are $G = 0$, $\frac{\partial G}{\partial n} = 0$, and $\frac{\partial G}{\partial n} + hG = 0$. For each region the admissible differential equations have also been characterized. It is interesting to note that of the problems which can be treated by the image method not all can be solved by separation of variables.

A generalization of the image method, namely inversion, applies to regions bounded by spheres but only works for Laplace's equation. It would be interesting to determine all the regions to which this method applies.

The image method can also be applied to differential equations on the surface of a sphere and to regions bounded by great circles. The present results yield all the regions on the sphere to which this method applies. We need only consider those Euclidean regions with one vertex, and for each construct a sphere with center at this vertex. The region cut out on the spherical surface by the Euclidean region is an admissible image region on the surface of the sphere, and conversely.

The method can also be extended to other surfaces, such as Riemann surfaces, as was done by Sommerfeld. In this case, however, the main difficulty is the construction of the Green's function for the full space.

Another extension is to regions bounded by an even number of pairwise parallel planes (or other surfaces) with the boundary condition of periodicity. This condition is that the boundary values of the solution must be the same

at corresponding points on two parallel congruent boundary planes. The admissible regions can be found by considering discrete groups of translations. If periodic boundary conditions are imposed on some planes, and $G = 0$ or $\frac{\partial G}{\partial n} = 0$ on others, then more general discrete groups, involving translations and reflections, will play a role. If periodic boundary conditions are imposed on two planes which meet at an angle, then discrete groups involving rotations must be considered.

For certain regions only a finite number of images occur, and therefore the sums used in defining the Green's function are finite. In other cases the convergence of the sum must be examined. This has been done in some cases. [2]

It is a pleasure to thank Professor Wilhelm Magnus for his helpful advice and criticism in connection with this paper.

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[1] Coxeter, H.S.M., Regular Polytopes, Methuen, London, 1948, Chapters 5 and 11.

[2] Sommerfeld, A., Partial Differential Equations in Physics, Vol I, Academic Press, New York, pp. 79-82.

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